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A NOTE ON ESSENTIAL NOETHERIAN MODULES AND RINGS

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Abstract. We recall that an $\mathcal{R}$-module $\mathcal{M}$ is essential Noetherian (in short, e-Noetherian) if any ascending chain of essential submodules becomes stationary. In this paper, we proved that for a class χ of $\mathcal{R}$-modules which is inherited by taking homomorphic images and extensions respectively, an e-Noetherian $\mathcal{R}$-module $\mathcal{M}$ fulfills ascending chain condition on χ-non-summands if and only if $\mathcal{M}/\mathcal{N}$ fulfills ascending chain condition on essential χ-submodules for each χ-non-summands $\mathcal{N}$ of $\mathcal{M}$. Also, we proved that every principal ideal ring $\mathcal{R}$ with ascending chain on essential principal left ideals is e-Noetherian.

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1. Introduction

In this paper, we consider the ring $\mathcal{R}$ as associative ring with unity and all modules are unital left $\mathcal{R}$-modules. A submodule $\mathcal{P}$ is essential if it has a non-trivial intersection with each non-trivial submodule of $\mathcal{M}$, that is, $\mathcal{P} \cap \mathcal{L} \neq 0$, for each non-trivial submodule $\mathcal{L}$ of $\mathcal{M}$. A module $\mathcal{M}$ is called uniform if every non-zero submodule is essential in $\mathcal{M}$. Socle of $\mathcal{M}$ is defined as the intersection of all essential submodules of $\mathcal{M}$. In [3], Osofsky introduced $\aleph$-chain on partially ordered set and studied the chain condition on essential submodules and it was demonstrated that for any infinite cardinal $\aleph$, an $\aleph$-chain condition on essential submodules of a module $\mathcal{M}$ is extremely near to that of $\aleph$-chain condition on all submodules. She also demonstrated that a module with the $\aleph$-chain condition on all submodules and a semisimple module directly add upto an $\mathcal{R}$ with the $\aleph$-chain condition on essential submodules. Also in [11], Dung et al. demonstrated that the ring $\mathrm{End}_{\mathcal{R}}(\mathcal{M})$ is a direct sum of a left Artinian ring and a (von Neumann) regular left-injective ring if $\mathcal{M}$ fulfills ascending chain condition on essential submodules and is a quasi-injective, quasi-projective left $\mathcal{R}$-module. In [13], authors had studied about the algebraic structure of e-Noetherian modules and e-Noetherian rings and they discussed several properties regarding these structures. In [7], the authors examined the structure of e-Artinian modules and rings and discussed various properties associated with these structures and also discussed descending chain condition on essential χ-submodules

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(summands and non-summands) of $\mathcal{M}$. Motivated by the above discussions, we study furthermore characterizations of e-Noetherian modules and rings also established several results in terms of χ-non-summands. Recall that an $\mathcal{R}$-module $\mathcal{M}$ is called quasi-injective module if for every submodule $\mathcal{N}$ of $\mathcal{M}$, any $\mathcal{R}$-homomorphism from $\mathcal{N}$ to $\mathcal{M}$ extends to an endomorphism of $\mathcal{M}$. An $\mathcal{R}$-module $\mathcal{M}$ is Noetherian if for any ascending chain of submodules, that is, $\mathcal{H}_{\mathbf{1}} \subseteq \mathcal{H}_{\mathbf{2}} \subseteq \mathcal{H}_{\mathbf{3}} \cdots \subseteq \mathcal{H}_{\boldsymbol{n}} \subseteq \cdots$ of $\mathcal{M}$ terminates, that is, there exists some natural number i such that $\mathcal{H}_{\boldsymbol{i}} = \mathcal{H}_{\boldsymbol{i+1}}$. For example, if we consider $\mathbb{Z}_{27}$, then the ascending chain $9\mathbb{Z}_{27} \subseteq 3\mathbb{Z}_{27} \subseteq \mathbb{Z}_{27}$ terminates. An $\mathcal{R}$-module $\mathcal{M}$ is semisimple (completely reducible) if each $\mathcal{R}$-submodule of $\mathcal{M}$ is a direct summand of $\mathcal{M}$. For more details on chain conditions on modules, refer to [2], [4], and [6].

This paper contains three sections where first section is about preliminary ideas and basic definitions. In the second section, we have studied about the characterizations of e-Noetherian modules. We proved that a module $\mathcal{M}$ is e-Noetherian if and only if $\mathcal{M}/Soc(\mathcal{M})$ is e-Noetherian (Theorem 2.1), and if $\mathcal{M}$ is an e-Noetherian $\mathcal{R}$-module satisfying ascending chain condition on finitely generated non-summands, then $\mathcal{M}/\mathcal{N}$ also satisfies the same for each fully invariant submodule $\mathcal{N}$ of $\mathcal{M}$ (Proposition 2.9). We also proved that if $\mathcal{M}$ is an uniserial module which satisfies the ascending chain condition on essential cyclic submodules, then $\mathcal{M}$ is a duo module (Theorem 2.14).
In the third section, we discuss about e-Noetherian rings, and we proved that every principal ideal ring $\mathcal{R}$ with an ascending chain of essential principal left ideal $\mathcal{A}_{\mathbf{1}} \subseteq \mathcal{A}_{\mathbf{2}} \subseteq \cdots \subseteq \mathcal{A}_{\boldsymbol{n}} \subseteq \cdots$ is e-Noetherian (Proposition 3.1). Also, if $\mathcal{R}$ is semi-prime e-Noetherian, we demonstrated that $\mathcal{R}/\mathcal{P}$ will be a left goldie ring for every minimal prime ideal $\mathcal{P}$ of $\mathcal{R}$ (Theorem 3.5). Besides, we demonstrated that if $\mathcal{R}$ is a ring with idempotent elements generating each maximal ideal, then $\mathcal{R}$ is e-Noetherian (Theorem 3.9).

We will refer [5], [14], [15] and [16] for all the basic terminologies and notations. In this paper, we consistently denote $J(\mathcal{R})$, $Soc(\mathcal{M})$, $Z({}_{\mathcal{R}}\mathcal{R})$, $E(\mathcal{R})$ as jacobson radical of a ring, socle of a module, singular ideal of ring, injective envelope of ring respectively.

2. Essential Noetherian Modules

An $\mathcal{R}$-module $\mathcal{M}$ is called an essential Noetherian (e-Noetherian) module if, for each essential ascending chain $\mathcal{L}_{\mathbf{1}} \subseteq_e \mathcal{L}_{\mathbf{2}} \subseteq_e \mathcal{L}_{\mathbf{3}} \subseteq_e \cdots$ of submodules of $\mathcal{M}$, there is a natural number m such that $\mathcal{L}_{\boldsymbol{i}} = \mathcal{L}_{\boldsymbol{m}}$ for any $i \geq m$. Clearly, each semisimple module, simple module, finite dimensional vector space, each uniform Noetherian module are an e-Noetherian. An example of e-Noetherian module is be $\mathbb{Z}_{81}$. If we consider the terminating chain $27\mathbb{Z}_{81} \subseteq_e 9\mathbb{Z}_{81} \subseteq_e 3\mathbb{Z}_{81} \subseteq_e \mathbb{Z}_{81}$, where every submodule in this chain is essential and thus $\mathbb{Z}_{81}$ is e-Noetherian.

Theorem 2.1. *An $\mathcal{R}$-module $\mathcal{M}$ is e-Noetherian if and only if $\mathcal{M}/Soc(\mathcal{M})$ is e-Noetherian.*

Proof. Let $\mathcal{M}$ be an e-Noetherian. Since $Soc(\mathcal{M})$ is a finite intersection of essential submodules and so $Soc(\mathcal{M})$ is an essential submodule of $\mathcal{M}$. Thus, $Soc(\mathcal{M})$ is e-Noetherian and hence we get the required result. Conversely, let us consider ascending chain $\mathcal{M}_{\boldsymbol{1}} \subseteq_e \mathcal{M}_{\boldsymbol{2}} \subseteq_e \cdots \subseteq_e \mathcal{M}_{\boldsymbol{n}} \subseteq_e \mathcal{M}_{\boldsymbol{n+1}} \subseteq_e \cdots$ of essential submodules of $\mathcal{M}$. Then $(\mathcal{M}_{\boldsymbol{1}} + Soc(\mathcal{M}))/Soc(\mathcal{M}) \subseteq_e (\mathcal{M}_{\boldsymbol{2}} + Soc(\mathcal{M}))/Soc(\mathcal{M}) \subseteq_e \cdots \subseteq_e (\mathcal{M}_{\boldsymbol{n}} + Soc(\mathcal{M}))/Soc(\mathcal{M}) \subseteq_e (\mathcal{M}_{\boldsymbol{n+1}} + Soc(\mathcal{M}))/Soc(\mathcal{M}) \subseteq_e \cdots$ is also an ascending chain of essential submodules in $\mathcal{M}/Soc(\mathcal{M})$. Since $\mathcal{M}/Soc(\mathcal{M})$ is e-Noetherian so there exists $n \in \mathbb{N}$, such that $(\mathcal{M}_{\boldsymbol{r+1}} + Soc(\mathcal{M}))/Soc(\mathcal{M}) = (\mathcal{M}_{\boldsymbol{r}} + Soc(\mathcal{M}))/Soc(\mathcal{M})$ for all $r \geq n$. Now, $(\mathcal{M}_{\boldsymbol{r}} + Soc(\mathcal{M}))/Soc(\mathcal{M}) \cong \mathcal{M}_{\boldsymbol{r}}$ for all r. Thus $\mathcal{M}_{\boldsymbol{r+1}} = \mathcal{M}_{\boldsymbol{r}}$, hence $\mathcal{M}$ is an e-Noetherian. □

Recall from [3], a poset $\mathcal{P}$ has the ascending $\aleph$-chain condition if and only if for each ordinal κ such that there exists a chain of subsets $\{\mathcal{N}_\alpha | \alpha < \kappa\}$ of $\mathcal{P}$ with $\mathcal{N}_\beta > \mathcal{N}_\alpha$ for all $\beta < \alpha$, we have $|\kappa| < \aleph$. Clearly Noetherian modules satisfy $\aleph_0$-chain condition.

Lemma 2.2. *If an e-Noetherian module satisfies $\aleph_0$-chain condition, then each submodule and quotient module satisfies $\aleph_0$-chain condition.*

Proof. Proof follows from [3, Lemma 1]. □

Theorem 2.3. *Let $\mathcal{M}$ be an e-Noetherian $\mathcal{R}$-module fulfilling ascending $\aleph_0$-chain condition, where $\aleph_0$ is the minimum cardinal such that*

(1) *There are less than $\aleph_0$ members in any family of independent submodules of $\mathcal{M}$.*
(2) *The set of all essential submodules of $\mathcal{M}$ has $\aleph_0$-chain condition.*

Then the collection of all submodules of $\mathcal{M}$ has an ascending $\aleph_0$-chain condition. Conversely, an $\aleph_0$-chain condition on each submodule of $\mathcal{M}$ gives (1) and (2).

Proof. Suppose $\{\mathcal{N}_\alpha | \alpha < \kappa\}$ be an ascending chain of distinct submodules of $\mathcal{M}$. We established a relationship " $\sim$ " on κ by $\alpha \sim \beta$ if and only if $\mathcal{N}_\alpha \subseteq_e \mathcal{N}_\beta$ or $\mathcal{N}_\beta \subseteq_e \mathcal{N}_\alpha$, which will be an equivalence relation. The set $\{\mathcal{N}_\alpha\}$ that corresponds to members of an equivalence class is an ascending chain of essential submodules in the union of the set. So we have by Lemma 2.2, every equivalence class has less than $\aleph_0$ members. Again, we have for every class contains a smallest member γ. Then, the well-order set $\rho = \{\sim -class\, representatives\, \gamma\}$ is a subchain of κ. Suppose μ denotes the successor of γ in ρ. Now, for each $\mu \in \rho$, the one which is smallest in $\{\mathcal{N}_\gamma, \mathcal{N}_\mu\}$ contains a closed submodule which is nonzero, that is, $\mathcal{L}_\gamma$ is the bigger one. These $\mathcal{L}_\gamma$ are independent and are a bijection with the sets of successors in ρ. If ρ is not finite, then its cardinality is equivalent to the set of successors. Hence, there must be less than $\aleph_0$ equivalence classes and since κ is a union of classes of items that are less than $\aleph_0$, it follows that $|\kappa| < \aleph_0$. The converse part is straightforward. □

Theorem 2.4. *Let $\mathcal{M}$ is an e-Noetherian module satisfying ascending $\aleph_0$-chain condition on essential submodules if and only if $\mathcal{M}/Soc(\mathcal{M})$ follows $\aleph_0$-chain condition on all submodules.*

Proof. Follows from [3, Theorem 3]. □

Recall from [12], a class χ of $\mathcal{R}$-modules is defined as collection of $\mathcal{R}$-modules that contains the zero module and it is closed under isomorphisms. A χ-module that is also a submodule (summand, non-summands) of $\mathcal{M}$ is referred to as a χ-submodule (in order to, χ-summand, χ-non-summads). A submodule $\mathcal{N}$ of a module $\mathcal{M}$ is called fully invariant in $\mathcal{M}$ if $g(\mathcal{N}) \subseteq \mathcal{N}$ for every endomorphism g of $\mathcal{M}$. If $\mathcal{N}$ is a fully invariant submodule of $\mathcal{M}$, then $\mathcal{N} = (\mathcal{N} \cap \mathcal{M}_{\mathbf{1}}) \oplus (\mathcal{N} \cap \mathcal{M}_{\mathbf{2}})$, consequently, $\mathcal{M}/\mathcal{N} = (\mathcal{M}_{\mathbf{1}} + \mathcal{N})/\mathcal{N} \oplus (\mathcal{M}_{\mathbf{2}} + \mathcal{N})/\mathcal{N}$ for all submodules $\mathcal{M}_{\mathbf{1}}$ and $\mathcal{M}_{\mathbf{2}}$ of $\mathcal{M}$ where $\mathcal{M} = \mathcal{M}_{\mathbf{1}} \oplus \mathcal{M}_{\mathbf{2}}$. The following results are related with chain condition on χ-non-summands of module.

Lemma 2.5. *If $\mathcal{M}$ is an e-Noetherian module fulfilling ascending chain condition on essential χ-non-summands, then so for every submodule of $\mathcal{M}$.*

Proof. Proof is similar to [12, lemma 2.3]. □

Proposition 2.6. *Suppose χ is a class of modules that is closed with respect to finite direct sums. If $\mathcal{M}$ is an $\mathcal{R}$-module fulfilling ascending chain condition on essential χ-non-summands and $\mathcal{P}$, $\mathcal{Q}$ be submodules of $\mathcal{M}$ such that $\mathcal{P} \cap \mathcal{Q} = 0$, then $\mathcal{P}$ has ascending chain condition on essential χ-submodules.*

Proof. Let $\mathcal{M}$ be an e-Noetherian module fulfilling ascending chain condition on χ-non-summands. By Lemma 2.5, $\mathcal{P} \oplus \mathcal{Q}$ fulfills ascending chain condition on χ-non-summands. Assume that we have a χ-submodule $\mathcal{K}$ of $\mathcal{Q}$ that is not a direct summands of $\mathcal{M}$. Let $\mathcal{H}_{\mathbf{1}} \subseteq_e \mathcal{H}_{\mathbf{2}} \subseteq_e \cdots$ be any ascending chain of χ-submodules of $\mathcal{P}$. For each $i \geq 1$, $\mathcal{H}_{\boldsymbol{i}} \cap \mathcal{K} = 0$ and $\mathcal{H}_{\boldsymbol{i}} \oplus \mathcal{K}$ is a χ-non-summands of $\mathcal{M}$. So we get $\mathcal{H}_{\mathbf{1}} \oplus \mathcal{K} \subseteq_e \mathcal{H}_{\mathbf{2}} \oplus \mathcal{K} \subseteq_e \cdots$ is an ascending chain of χ-non-summands of $\mathcal{M}$ and hence $\mathcal{H}_{\boldsymbol{n}} \oplus \mathcal{K} = \mathcal{H}_{\boldsymbol{n+1}} \oplus \mathcal{K} = \cdots$ for finite n. Hence, $\mathcal{P}$ fulfills ascending chain condition on χ-submodules. □

Theorem 2.7. *Let χ denote a class of $\mathcal{R}$-modules that are closed under direct summands and finite direct sums respectively, then an e-Noetherian module $\mathcal{M}$ fulfilling ascending chain condition on χ-non-summands if and only if each χ-non-summands of $\mathcal{M}$ fulfills ascending chain condition on essential χ-submodules.*

Proof. We note that the sufficiency part is obvious. Conversely, let $\mathcal{M}$ be an e-Noetherian module fulfilling ascending chain condition on χ-non-summands. Suppose that H is a χ-submodule of $\mathcal{M}$ such that there exists a proper ascending chain $\mathcal{H} = \mathcal{H}_{\mathbf{1}} \subset_e \mathcal{H}_{\mathbf{2}} \subset_e \cdots$ of χ-non-summands of $\mathcal{M}$. By hypothesis, there exists a natural number n such that $\mathcal{H}_{\boldsymbol{n}}$ is a direct summand of $\mathcal{M}$. Suppose $\mathcal{K}$ is a submodule of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{H}_{\boldsymbol{n}} \oplus \mathcal{K}$. For every $i \geq n$, $\mathcal{H}_{\boldsymbol{i}} = \mathcal{H}_{\boldsymbol{n}} \oplus (\mathcal{H}_{\boldsymbol{i}} \cap \mathcal{K})$. By hypothesis, there is a properly ascending chain $(\mathcal{H}_{\boldsymbol{n}} \cap \mathcal{K}) \subset_e (\mathcal{H}_{\boldsymbol{n+1}} \cap \mathcal{K}) \subset_e \ldots$ of χ-submodule of $\mathcal{K}$. By Proposition 2.6, $\mathcal{H}$ is a direct summand of $\mathcal{M}$. Thus, the proof follows. □

Proposition 2.8. *Let χ be a class of $\mathcal{R}$-modules which is inherited by taking homomorphic images and extensions, respectively. Then $\mathcal{M}$ is an e-Noetherian $\mathcal{R}$-module fulfills ascending chain condition on essential χ-non-summands if and only if $\mathcal{M}/\mathcal{N}$ fulfills ascending chain condition on essential χ-submodules for each χ-non-summands $\mathcal{N}$ of $\mathcal{M}$.*

Proof. Suppose $\mathcal{M}$ fulfills ascending chain condition on essential χ-non-summands of $\mathcal{M}$. Assume $\overline{\mathcal{P}_{\mathbf{1}}} \subseteq_e \overline{\mathcal{P}_{\mathbf{2}}} \subseteq_e \cdots$ be an ascending chain condition of essential essential χ-submodules of $\mathcal{M}/\mathcal{N}$. For every $i \geq 1$, $\overline{\mathcal{P}_{\boldsymbol{i}}} = \mathcal{P}_{\boldsymbol{i}}/\mathcal{N}$ for some arbitrary submodule $\mathcal{P}_{\boldsymbol{i}}$ of $\mathcal{M}$, which contains $\mathcal{N}$. By hypothesis, $\mathcal{P}_{\boldsymbol{i}}$ is a χ-submodule of $\mathcal{M}$ for all $i \geq 1$. By Theorem 2.7, $\mathcal{P}_{\boldsymbol{n}} = \mathcal{P}_{\boldsymbol{n+1}} = \cdots$ for finite $n \in \mathbb{N}$, then $\mathcal{P}_{\boldsymbol{n}}/\mathcal{N} = \mathcal{P}_{\boldsymbol{n+1}}/\mathcal{N} = \cdots$ for some natural nunber n. Thus $\mathcal{M}/\mathcal{N}$ satisfies the ascending chain condition on essential χ-submodules. Conversely, assume that $\mathcal{M}/\mathcal{N}$ fulfills ascending chain condition on essential χ-submodules for every χ-non-summands $\mathcal{N}$ of $\mathcal{M}$. Let $\mathcal{P}$ be any χ-non-summands of $\mathcal{M}$ and let $\mathcal{K}_{\mathbf{1}} \subseteq_e \mathcal{K}_{\mathbf{2}} \subseteq_e \cdots$ be any ascending chain of χ-submodules of $\mathcal{M}$ such that $\mathcal{P} \subseteq \mathcal{K}_{\mathbf{1}}$. Then $\mathcal{K}_{\mathbf{1}}/\mathcal{P} \subseteq_e \mathcal{K}_{\mathbf{2}}/\mathcal{P} \subseteq_e \cdots$ is an ascending chain of χ-submodules of $\mathcal{M}/\mathcal{P}$. There exists a natural number r such that $\mathcal{K}_{\boldsymbol{r}}/\mathcal{P} = \mathcal{K}_{\boldsymbol{r+1}}/\mathcal{P} = \cdots$ and obviously $\mathcal{K}_{\boldsymbol{r}} = \mathcal{K}_{\boldsymbol{r+1}} = \cdots$. By Theorem 2.7, $\mathcal{M}$ fulfills ascending chain condition on essential χ-non-summands. □

Proposition 2.9. *If $\mathcal{M}$ is an e-Noetherian $\mathcal{R}$-module satisfying ascending chain condition on finitely generated non-summands, then $\mathcal{M}/\mathcal{N}$ also satisfies the same for each fully invariant submodule $\mathcal{N}$ of $\mathcal{M}$.*

Proof. Assume that $\mathcal{N}$ is a fully invariant non-zero submodule of $\mathcal{M}$. Let $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{N}$ and $\overline{\mathcal{P}} \subseteq \overline{\mathcal{H}}$ be finitely generated submodules of $\overline{\mathcal{M}}$. Now there exists natural number r, s and elements a_i, b_j in $\mathcal{M}$ $(1 \leq i \leq r, 1 \leq j \leq s)$ such that $\overline{\mathcal{H}} = (a_1 + \mathcal{N})\mathcal{R} + \cdots + (a_r + \mathcal{N})\mathcal{R}$ and $\overline{P} = (b_1 + \mathcal{N})\mathcal{R} + \cdots + (b_s + \mathcal{N})\mathcal{R}$. For each $1 \leq j \leq s$ there exists elements $r_{ij} \in \mathcal{R}$ $(1 \leq i \leq r)$ and $u_j \in \mathcal{N}$ such that $b_j = \sum_{i=1}^{r} b_j r_{ij} + u_j$. Let $z_j = \sum_{i=1}^{r} a_i r_{ij}$ $(1 \leq j \leq s)$. Then $\overline{\mathcal{P}} = (z_1+\mathcal{N})\mathcal{R}+\cdots+(z_t+\mathcal{N})\mathcal{R}$ and $z_1\mathcal{R}+\cdots+z_t\mathcal{R} \subseteq a_1\mathcal{R}+\cdots+a_s\mathcal{R}$. Assume that $\overline{\mathcal{P}_{\mathbf{1}}} \subseteq_e \overline{\mathcal{P}_{\mathbf{2}}} \subseteq_e \cdots$ represents any ascending chain of finitely generated non-summands of $\overline{\mathcal{M}}$. We can assume without loss of generality, that $\overline{\mathcal{P}_{\boldsymbol{i}}} = (\mathcal{P}_{\boldsymbol{i}} + \mathcal{N})/\mathcal{N}$ for some ascending chain $\mathcal{P}_{\mathbf{1}} \subseteq \mathcal{P}_{\mathbf{2}} \subseteq \cdots$ of finitely generated submodules of $\mathcal{M}$. Also, we have $\mathcal{P}_{\boldsymbol{i}}$ is a non-summands of $\mathcal{M}$ for each $i \geq 1$ and by given statement, there is always a natural number k fulfills $\mathcal{P}_{\boldsymbol{k}} = \mathcal{P}_{\boldsymbol{k+1}} = \cdots$ and thus $\overline{\mathcal{P}_{\boldsymbol{k}}} = \overline{\mathcal{P}_{\boldsymbol{k+1}}} = \cdots$ □

Recall from [16], a ring $\mathcal{R}$ is said to be semiprimary if $\mathcal{R}/J(\mathcal{R})$ is semisimple and $J(\mathcal{R})$ is nilpotent. An ideal $\mathcal{I}$ of a ring $\mathcal{R}$ is considered as left T-nilpotent if for every sequence $\{x_n\}$ of elements in $\mathcal{I}$ there is a natural number n such that $x_n x_{n-1} \cdots x_3 x_2 x_1 = 0$ and for $\mathcal{S} = End_{\mathcal{R}}(\mathcal{R})$ define $\mathcal{Z} = \{\psi \in \mathcal{S} : Ker(\psi)$ is an essential $\mathcal{R}$-submodule of $\mathcal{M}={}_{\mathcal{R}}\mathcal{R}\}$.

Lemma 2.10. *If $\mathcal{M}$ is an e-Noetherian $\mathcal{R}$-module, then the ascending chain condition on essential left annihilators is fulfilled by $\mathcal{S} = End_{\mathcal{R}}(\mathcal{M})$.*

Proof. Proof is dual of [9, Lemma 1.1] □

Lemma 2.11. *Let $\mathcal{S}$ be a ring which fulfills the ascending chain condition on essential left annihilator. Then a subring $\mathcal{N}$ of $\mathcal{S}$ is T-nilpotent if and only if $\mathcal{N}$ is left nilpotent.*

Proof. Proof is similar to [10, Proposition 1.5]. □

Proposition 2.12. *If $\mathcal{M}$ is an e-Noetherian left $\mathcal{R}$-module, then $\mathcal{Z}$ is nilpotent.*

Proof. Since $\mathcal{S}$ fulfills ascending chain condition on essential left annihilators by Lemma 2.10, it is sufficient to prove that $\mathcal{Z}$ is left T-nilpotent. Let $\{\psi_n\}$ be a sequence of elements in $\mathcal{Z}$. Set $x_j = \psi_j\psi_{j-1}\cdots\psi_2\psi_1$. Then $Ker(x_1) \subseteq_e Ker(x_2) \subseteq_e \cdots \subseteq_e Ker(x_j) \subseteq_e \cdots$ is an ascending chain condition of essential $\mathcal{R}$-submodules of $\mathcal{M}$. Since $\mathcal{M}$ is e-Noetherian then there exists a natural number $n \in \mathbb{N}$ such that $Ker(x_k) = Ker(x_{k+1}) = \cdots$. Let $y \in Im(x_k) \cap Ker(\psi_{k+1})$ this implies that $y \in Im(x_k)$ and $y \in Ker(\psi_{k+1})$. Since $Im(x_k) \subseteq_e Ker(\psi_{k+1})$ follows from above chain and $Im(x_k) \cap Ker(\psi_k) = \{0\}$, implies that $y = 0$. Thus, $Im(x_k) \cap Ker(\psi_{k+1}) = \{0\}$. Therefore $(\mathcal{M})x_k \cap Ker(\psi_{k+1}) = \{0\}$. Since $Ker(\psi_{k+1})$ is an essential $\mathcal{R}$-submodule of $\mathcal{M}$, it follows that $(\mathcal{M})x_k = 0$. Thus, $x_k = \psi_k\psi_{k-1}\cdots\psi_2\psi_1 = 0$. Therefore $\mathcal{Z}$ is left T-nilpotent, by Lemma 2.11, $\mathcal{Z}$ is nilpotent. □

Corollary 2.13. *If $\mathcal{M}$ is quasi injective e-Noetherian $\mathcal{R}$-module, then $\mathcal{S} = End_{\mathcal{R}}(\mathcal{M})$ is semiprimary.*

Proof. According to [16, Proposition 2], $\mathcal{S}$ is semiperfect and $\mathcal{Z}$ is the Jacobson radical of $End_{\mathcal{R}}(\mathcal{M})$. By Proposition 2.12, $\mathcal{Z}$ is T-nilpotent then it follows that $End_{\mathcal{R}}(\mathcal{M})$ is semiprimary. □

Recall from [1], a module $\mathcal{M}$ is termed as uniserial if for any submodules $\mathcal{L}$ and $\mathcal{N}$ of $\mathcal{M}$, it holds that either $\mathcal{L} \subseteq \mathcal{N}$ or $\mathcal{N} \subseteq \mathcal{L}$. A submodule N of M is called an *M-cyclic submodule* if there exists a submodule L of M such that $N \cong M/L$ (see [17]). .

Theorem 2.14. *If $\mathcal{M}$ fulfills the ascending chain condition on essential $\mathcal{M}$-cyclic submodules and also be an uniserial module, then $\mathcal{M}$ is a duo module.*

Proof. Assume that $\mathcal{M}$ satisfies the ascending chain condition on essential $\mathcal{M}$-cyclic submodules. Let $\mathfrak{m} \neq 0$ be an element of $\mathcal{M}$, and let $\mathfrak{f}$ be an endomorphism of $\mathcal{M}$. Suppose that $\mathfrak{f}(\mathfrak{m}) \notin \mathfrak{m}\mathcal{R}$, then $\mathfrak{m} \in \mathfrak{f}(\mathfrak{m})\mathcal{R}$ that is, there exists some $r \in \mathcal{R}$ such that $\mathfrak{m} = \mathfrak{f}(\mathfrak{m})r$. It follows that for every positive integer n such that $\mathfrak{f}^{\mathfrak{n}}(\mathfrak{m}) = \mathfrak{f}^{\mathfrak{n}+1}(\mathfrak{m})r$. Thus, any ascending chain $\mathfrak{m}\mathcal{R} \subseteq_e \mathfrak{f}(\mathfrak{m})\mathcal{R} \subseteq_e \mathfrak{f}^2(\mathfrak{m})\mathcal{R} \subseteq_e \cdots$ of essential $\mathcal{M}$-cyclic submodules of $\mathcal{M}$. By hypothesis, there is a positive number k such that $\mathfrak{f}^k(\mathfrak{m})\mathcal{R} = \mathfrak{f}^{k+1}(\mathfrak{m})\mathcal{R}$. There exists $s \in \mathcal{R}$ such that $\mathfrak{f}^{k+1}(\mathfrak{m}) = \mathfrak{f}^k(\mathfrak{m})s = \mathfrak{f}^k(\mathfrak{m}s)$. This implies that $\mathfrak{f}^k(\mathfrak{f}(\mathfrak{m}) - \mathfrak{m}s) = 0$ that is, $\mathfrak{f}(\mathfrak{m}) - \mathfrak{m}s \in ker(\mathfrak{f}^k)$. If $\mathfrak{m}\mathcal{R} \subseteq_e ker(\mathfrak{f}^k)$, then $\mathfrak{f}^k(\mathfrak{m}) = 0$, which would lead to $\mathfrak{m} = \mathfrak{f}^k(\mathfrak{m})r^k = 0$, a contradiction. Thus, $ker(\mathfrak{f}^k) \subseteq_e \mathfrak{m}\mathcal{R}$ and consequently, $\mathfrak{f}(\mathfrak{m}) - \mathfrak{m}s \in \mathfrak{m}\mathcal{R}$ this implies that $\mathfrak{f}(\mathfrak{m}) \in \mathfrak{m}\mathcal{R}$, a contradiction. Therefore, $\mathfrak{f}(\mathfrak{m}) \in \mathfrak{m}\mathcal{R}$, by [1, lemma 1.1], $\mathcal{M}$ is a duo module. □

3. Essential Noetherian rings

A ring $\mathcal{R}$ is left essential Noetherian (in short, e-Noetherian) if the module ${}_{\mathcal{R}}\mathcal{R}$ is e-Noetherian. Equivalently, we can say that a ring $\mathcal{R}$ fulfills ascending chain condition on essential left ideals of $\mathcal{R}$, that is, for every essential ascending chain $\mathcal{I}_{\mathbf{1}} \subseteq_e \mathcal{I}_{\mathbf{2}} \subseteq_e \mathcal{I}_{\mathbf{3}} \subseteq_e \cdots$ of left ideals of $\mathcal{R}$, there exists a number $n \in \mathbb{N}$ such that $\mathcal{I}_{\boldsymbol{n}} = \mathcal{I}_{\boldsymbol{j}}$ for all $j \geq n$. For example division rings, finite rings are all e-Noetherian. If there is a finite subset $\mathcal{X}$ of $\mathcal{R}$ such that $\mathcal{I} =< \mathcal{X} >$, then an ideal $\mathcal{I}$ of $\mathcal{R}$ is finitely generated. If an ideal $\mathcal{I}$ is generated by one element, then it is known as principal ideal. A principal ideal ring is a ring $\mathcal{R}$ where each ideal is principal. If in addition to $\mathcal{R}$ is also an integral domain, it is known as the principal ideal domain.

Proposition 3.1. *A principal ideal ring $\mathcal{R}$ with ascending chain of essential principal left ideals $\mathcal{A}_{\mathbf{1}} \subseteq \mathcal{A}_{\mathbf{2}} \subseteq \cdots \subseteq \mathcal{A}_{\boldsymbol{n}} \subseteq \cdots$ is e-Noetherian.*

Proof. Consider the family of essential principal left ideal of $\mathcal{R}$ as $\{\mathcal{A}_{\boldsymbol{i}} : i \in \mathbb{N}\}$ such that $\mathcal{A}_{\boldsymbol{r}} \subset \mathcal{A}_{\boldsymbol{r+1}}$ for all $r \in \mathbb{N}$. Then $\mathcal{A} = \bigcup_{i\in\mathbb{N}} \mathcal{A}_{\boldsymbol{i}}$ is also essential principal left ideal of $\mathcal{R}$, as each $\mathcal{A}_{\boldsymbol{i}}$ is essential in $\mathcal{R}$. Let $\mathcal{A}$ be generated by an element $p \in \mathcal{A}$. Now, since $p \in \mathcal{A}$, there exists an index $k \in \mathbb{N}$ such that $p \in \mathcal{A}_{\boldsymbol{k}}$.
We claim that $\mathcal{A}_{\boldsymbol{k}} = \mathcal{A}_{\boldsymbol{r}}$ for all $r \geq k$. Suppose it is not true, then there exists $r \geq k$ such that $\mathcal{A}_{\boldsymbol{k}} \subset \mathcal{A}_{\boldsymbol{r}}$ and $\mathcal{A}_{\boldsymbol{k}} \neq \mathcal{A}_{\boldsymbol{r}}$ that is, $\mathcal{A}_{\boldsymbol{r}} \setminus \mathcal{A}_{\boldsymbol{k}}$ is nonempty. Consider $x \in \mathcal{A}_{\boldsymbol{r}}$ but $x \notin \mathcal{A}_{\boldsymbol{k}}$, then $x \in \mathcal{A} = \bigcup_{i\in\mathbb{N}} \mathcal{A}_{\boldsymbol{i}}$, so $x = qp$ for some $q \in \mathcal{R}$ as p is a generator of $\mathcal{A}$. Also, since $\mathcal{A}_{\boldsymbol{k}}$ is left ideal and $p \in \mathcal{A}_{\boldsymbol{k}}$, we have $qp \in \mathcal{A}_{\boldsymbol{k}}$ as $x = qp$ implies that $x \in \mathcal{A}_{\boldsymbol{k}}$ which arrives at contradiction to our supposition. Thus the given chain of essential principal left ideal terminates. Hence $\mathcal{R}$ is e-Noetherian. □

Corollary 3.2. *Every principal ideal domain with essential ideal is e-Noetherian.*

Proof. The proof follows from the above proposition. □

Proposition 3.3. *Assume that $\mathcal{R}$ is a commutative ring, then we have*

(1) *If $\mathcal{R}$ is e-Noetherian, then it fulfills the descending chain condition on annihilators.*
(2) *If $\mathcal{R}$ be self-injective, then $\mathcal{R}$ is e-Noetherian if and only if $\mathcal{R}$ is Noetherian.*

Proof. (1) Let $\mathcal{A}_{\mathbf{1}} \supseteq_e \mathcal{A}_{\mathbf{2}} \supseteq_e \cdots$ be a descending chain of an ideals of $\mathcal{R}$ that are annihilators of subsets of $\mathcal{R}$. If we take annihilators, then we obtain an ascending condition $ann(\mathcal{A}_{\mathbf{1}}) \subseteq_e ann(\mathcal{A}_{\mathbf{2}}) \subseteq_e \cdots$. Because $\mathcal{R}$ is e-Noetherian, then there is a natural number n such that $ann(\mathcal{A}_{\boldsymbol{i}}) = ann(\mathcal{A}_{\boldsymbol{n}})$ for all $i \geq n$. Again, considering annihilators of the annihilators ideals, we obtain $\mathcal{A}_{\boldsymbol{i}} = \mathcal{A}_{\boldsymbol{n}}$ for each $i \geq n$.
(2) Because $\mathcal{R}$ is self-injective and e-Noetherian, by the preceding argument, $\mathcal{R}$ holds descending chain condition on annihilators and then by using [8, Theorem 2], $\mathcal{R}$ is Quasi-Frobenius hence $\mathcal{R}$ is Noetherian. Converse is trivial.

□

Recall from [16], an ideal $\mathcal{I}$ of $\mathcal{R}$ is semi-prime if and only if $\mathcal{A}^2 \subseteq \mathcal{I}$ implies $\mathcal{A} \subseteq \mathcal{I}$. If $\{0\}$ is the only semi-prime ideal, a ring $\mathcal{R}$ is semi-prime. If a ring $\mathcal{R}$ fulfills the ascending chain condition on left annihilators and $u.dim({}_{\mathcal{R}}\mathcal{R}) < \infty$, then it is left Goldie. If every element of an ideal $\mathcal{I}$ of a ring $\mathcal{R}$ is nilpotent, the ideal $\mathcal{I}$ is considered to be nil.

Theorem 3.4. *A semi-prime left e-Noetherian ring $\mathcal{R}$ is a Goldie ring.*

Proof. Let $\mathcal{R}$ be an left e-Noetherian ring. Then $\mathcal{R}$ satifies descending chain condition on annihilators by Proposition 3.3(1) after taking annihilators. Again, by [16, Theorem 11.43], we have $\mathcal{R}$ has $u.dim(\mathcal{R}) < \infty$. Thus, $\mathcal{R}$ is a Goldie ring. □

Theorem 3.5. *Let $\mathcal{R}$ be a semi-prime left e-Noetherian ring, then for each simple prime ideal $\mathcal{P}$ of $\mathcal{R}$, the quotient $\mathcal{R}/\mathcal{P}$ forms a Goldie ring.*

Proof. Assume $\mathcal{R}$ is a semi-prime ring. As $\mathcal{R}$ is left e-Noetherian, it fulfills a descending chain condition on the annihilator ideals which implies that $\mathcal{R}$ has a limited number of simple prime ideals. According to the previous Theorem 3.4, $\mathcal{R}$ is a Goldie ring and thus, by [16, Corollary 11.44], for each simple prime ideal of $\mathcal{R}$, $\mathcal{R}/\mathcal{P}$ is a Goldie ring. □

Proposition 3.6. *If $\mathcal{R}$ is an e-Noetherian ring, then $\mathcal{Z}({}_{\mathcal{R}}\mathcal{R})$ is a nil ideal, where $\mathcal{Z}({}_{\mathcal{R}}\mathcal{R})$ is a singular ideal.*

Proof. Let $y \in \mathcal{Z}({}_{\mathcal{R}}\mathcal{R})$ and since $\mathcal{R}$ is e-Notherian so we have $ann(y^m) = ann(y^{m+1})$ for some $m \geq 1$. We claim that $y^m = 0$. Assume the contrary of the statement that $y^m \neq 0$. Then $ann_(y^m) \cap \mathcal{R}y^m$ will have a non-zero element py^m for some $p \in \mathcal{R}$. Again, since $py^m y^m = 0$, because $py^m \in ann(y^m)$. Thus we get $p \in ann(y^{2m}) = ann(y^m)$. This implies $py^m = 0$, which is a contradiction. Hence $Z({}_{\mathcal{R}}\mathcal{R})$ must be a nil ideal. □

We recall that if $\mathcal{R}$ is a ring, $E(\mathcal{R})$ be an injective envelope of $\mathcal{R}$ and $\mathcal{H} = Hom_{\mathcal{R}}(E(\mathcal{R}), E(\mathcal{R}))$ and so we can attain a bimodule $E(\mathcal{R})$. Let $\mathcal{Q} = Hom_{\mathcal{H}}(E(\mathcal{R}), E(\mathcal{R}))$ where $\mathcal{Q}$ is known as maximal left quotient ring of $\mathcal{R}$.

Theorem 3.7. *Let $\mathcal{R}$ be a non-singular ring with maximal quotient ring $\mathcal{Q}$. If $\mathcal{M}$ is a left $\mathcal{Q}$-module that is a non-singular left $\mathcal{R}$-module and e-Noetherian, then $\mathcal{M}$ is an e-Noetherian as a $\mathcal{Q}$-module.*

Proof. Suppose $\mathcal{K}_{\mathbf{1}} \subseteq_e \mathcal{K}_{\mathbf{2}} \subseteq_e \cdots$ be an ascending chain of essential $\mathcal{Q}$-submodules of $\mathcal{M}$. Clearly, this is also an ascending chain of $\mathcal{R}$-submodules of $\mathcal{M}$ and therefore, for some n, $\mathcal{K}_{\boldsymbol{n}} = \mathcal{K}_{\boldsymbol{i}}$ as $\mathcal{R}$-modules for each $i \geq n$. Let ϕ_i be an endomorphism of $\mathcal{R}$ if $q \in \mathcal{Q}$, so we have an essential left ideal E of $\mathcal{R}$ such that $Eq \subseteq_e \mathcal{R}$. Then, for every $t \in \mathcal{K}_{\boldsymbol{n}}$ and $e \in E$, $\phi_i(eqt) = e\phi_i(qt)$ and $\phi_i(eqt) = eq\phi_i(t)$, so $e(\phi_i(qt) - q\phi_i(t)) = 0$ that is, $\mathcal{R}(\phi_i(qt) - q\phi_i(t)) = 0$ and because $\mathcal{M}$ is non-singular, we get $\phi_i(qt) = q\phi_i(t)$. Hence, ϕ_i is a $\mathcal{Q}$-endomorphism. □

Proposition 3.8. *Assuming $\mathcal{R}$ is a regular ring, $\mathcal{R}$ is e-Noetherian if and only if it is semisimple.*

Proof. Since $\mathcal{R}$ is e-Noetherian, we have each ideal to be finitely generated and since $\mathcal{R}$ being regular means that each finitely generated ideal is a direct summand of $\mathcal{R}$, which implies that $\mathcal{R}$ is semisimple. Converse part is obvious. □

We recall that an ideal $\mathcal{A}$ is said to be primary ideal of $\mathcal{R}$ if whenever $xy \in \mathcal{A}$ for all $x, y \in \mathcal{R}$, then either $x \in \mathcal{A}$ or $y^n \in \mathcal{A}$ for some natural number n.

Theorem 3.9. *If $\mathcal{R}$ be a ring with idempotent elements generating each maximal ideal, then $\mathcal{R}$ is an e-Noetherian ring.*

Proof. Let $\mathcal{A}$ be a primary ideal of $\mathcal{R}$. We claim to show that $\mathcal{A}$ is a maximal ideal otherwise, there exists a maximal ideal m such that $\mathcal{A} \subsetneq m$. Then, by hypothesis, $m =< e >$ where e is an idempotent element in $\mathcal{R}$ such that $e \neq 0$ or $e \neq 1$, since $e \neq 0$ implies that $\mathcal{R}$ is a field and the proof is trivial. Then $e(1-e) = 0 \in \mathcal{A}$, and $\mathcal{A}$ is a primary ideal. Thus $(1-e)^n \in \mathcal{A} \subsetneq m$ for some positive integer n, implies that $1 - e \in m =< e >$ gives $1 \in m$ which is a contradiction. Thus $\mathcal{A}$ is a maximal ideal. Since every primary ideal of $\mathcal{R}$ is maximal, the concepts maximal, prime and primary ideals coincide. Hence, by our hypothesis, every maximal ideals is finitely generated. Hence $\mathcal{R}$ is e-Noetherian. □

References

[1] A. C. Ozcan, A. Harmanci and P. F. Smith, *Duo Modules*, Glasg. Math .J., 48:533–545, 2006, doi:10.1017/S0017089506003260.

[2] A. Tercan and C. C. Yücel, *Extending Modules and Generalizations*, Springer, 2016.

[3] B. L. Osofsky, *Chain conditions on essential submodules*, Proc. Amer. Math. Soc., 114(1):11-19, 1992.

[4] C. Procesi and L. Small, *Endomorphism rings of modules overs PI-Algebra*, Math Z, 106:178-180, 1968, doi:10.1007/BF01110128.

[5] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verl., 1992.

[6] H. Chakraborty, M. K. Patel and R. K. Singh, Acc on essential residuals and colon ideals, Asian-European Journal of Mathematics, 2550086(1-11), 2025, doi: 10.1142/S179355712550086X.

[7] H. Chakraborty, M. K. Patel and R. K. Singh, Some Aspects On Essential Artinian Modules and Rings, Asian-European Journal of Mathematics, 2025, doi: 10.1142/S1793557125501013.

[8] J. Clark, *When is a Self-Injective Semi Perfect Ring Quasi-Frobenius?*, J. Algebra, 165:531-542, 1994, doi:10.1006/jabr.1994.1128.

[9] J. W. Fisher, *Nil subrings of endomorphism rings of modules*, Proc. Amer. Math. Soc., 34:75-78, 1972.

[10] J. W. Fisher, *On the nilpotency of nil subrings*, Can. J. Math., 22(6):1211-1216, 1970.

[11] N. V. Dung, D. V. Huynh and R. Wisbauer, *Quasi-injective modules with acc or dcc on essential submodules*, Arch. Math., 53:252–255, 1989, doi:10.1007/BF01277059.

[12] A. Ç. Özcan and P. F. Smith, *Chain conditions on non-summands*, J. Algebra Appl., 10(03):475-489, 2011, doi:10.1142/S0219498811004707.

[13] P. Nath, H. Chakraborty and G. Bardhan, *e-Noetherian modules and rings*, De Gruyter proceeding on Advances in Pure and Applied Algebra, Berlin, Boston, 101–108, 2023, doi:10.1515/9783110785807-013.

[14] R. B. Warfield Jr. and K. R. Goodearl, *An Introduction to Noncommutative Noetherian Rings*, Cambridge Univ. Press, 2004.

[15] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon Breach Sci. Publ., Routledge, 1988, doi:10.1201/9780203755532.

[16] T. Y. Lam, *Lectures on Modules and Ring*, Springer, New York, 1998.

[17] V. Kumar, A. J. Gupta, B. M. Pandeya and M. K. Patel, M-sp-injective modules, Asian European Journal of Mathematics, 5(1), 1250005 (1-11), 2012, doi: 10.1142/S1793557112500052.

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